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# Technical Documentation for SFINCS with multiple species

## Introduction

In this document, we detail the equations implemented in SFINCS: the Stellarator Fokker-Planck Iterative Neoclassical Conservative Solver. The normalizations and input and output quantities are also defined.

## Kinetic equation

We begin with the following drift-kinetic equation:



where  denotes species, ,



is the total energy,  is the charge in units of the proton charge ,  is the temperature,  is the mass,  is the  drift,



is the magnetic drift,  is the gyrofrequency (which is negative for electrons with ), and  is the speed of light. Subscripts on partial derivatives indicate quantities held fixed in differentiation. We assume the electrostatic potential  is a flux function to the order of interest. The total distribution function is  where

.

In ,  is the collision operator linearized about the Maxwellian  for each species. We neglect contributions from the inductive electric field to , writing  where  is the toroidal flux. Let  be the thermal speed, and let .

The independent variables used in SFINCS are  where . Changing velocity variables to  on the left side of ,



where

,

,

and

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Applying  to we find

,

so simplifies to

.

Similarly, applying  to , we find

.

Thus, may be written

.

Noting



then the two electric field terms in may be combined to give

.

In the present implementation of SFINCS, the  terms in and are neglected, as is the  term in . (This last term must be dropped in order to maintain conservation of .) We are then left with

,

,

.

These are the same terms as in the last section of the appendix of Ref. [1].

We can verify that - still conserve :



As shown in the appendix of Ref. [1], - do not conserve  because the radial magnetic drift has been dropped. However, in an axisymmetric or quasisymmetric field, - do conserve a combination of energy and canonical momentum.

To compare various effective particle trajectories, the code allows the  terms in and to be turned off, in which case

,

.

For comparison with DKES, SFINCS allows the option of using

,

in place of .

One further option allowed in the code is to also include a term



on the left-hand side of . The rationale for including this term is that it allows the left-hand side of to be put into a conservative form when - are used:

.

For the rest of these notes, we will include the term multiplied by , so  will be either 0 or 1.

Now consider the magnetic field in Boozer coordinates:

,

where  is the rotational transform with  the safety factor, and

,

where , ,  is the poloidal current outside the flux surface, and  is the toroidal current inside the flux surface. Notice . The product of with gives the Jacobian

.

Notice also that



for any quantity , and

.

The kinetic equation with - is thus equivalent to



where

,

,

,

.

## Normalizations

Let’s suppose we are given , , , , , , , , , , and  where  is the normalized toroidal flux. The flux at the last closed flux surface is , so the dimensional flux  is related to  by . The input quantities are specified in terms of some species-independent dimensions  (e.g. eV),  (e.g. 1020/m3),  (e.g. kV),  (e.g. T),  (e.g. m), and  (typically the proton or deuteron mass). In other words, the quantities we are actually given are

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and

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Notice , and so



for any flux function .

It will be useful to define the following combinations of normalization constants:

,



(which resembles ),

,

,

and a normalized collisionality



where  is the dimensional collisionality at the reference parameters:

.

We assume  has the same value for all species. It will be useful to notice

.

We define a normalized distribution function  as follows:

.

Notice the normalization is the same for each species. Also notice that the normalization is different than in the original 1-species version of SFINCS.

The kinetic equation for each species is made dimensionless by multiplying through by

.

(this normalization too is slightly different than in the 1-species version of SFINCS.) We then obtain



where .

## Legendre discretization

SFINCS uses a collocation discretization in the , , and  coordinates, but a modal discretization in the  coordinate. In other words, the distribution function is known at certain grid points in , , and , but it is expanded as modes in . We employ the following modal expansion in terms of Legendre polynomials :

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We discretize the kinetic equation by applying

.

To evaluate the various integrals that result, the following identities may be used:

,



,



,

and

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As a result, may be written



where

,







,

,

and



## Collision operator

The total collision operator for species  is a sum of collision operators with each species:

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The linearized Fokker-Planck collision operator for each pair of species may be written

,

where the Lorentz part of the collision term is



with

,

,

.

The energy scattering contribution is



where

.

The field term is



where the potentials are defined by



and

.

We write the field term as



where







The Poisson equations that define the potentials are (for Legendre mode )



.

Let us define





so the defining equations for the potentials become



.

Next, recall that in the kinetic equation , we need to evaluate



where  is defined in . It is convenient to note



Expanding  as before,



where



The energy scattering component is



The diagonal term is



In the cross-species case, this term is no longer identical to the  term in energy scattering (as it is in the same-species case).

The  term in the collision operator is



Although in principle we would also be free to write



(i.e. replacing  in two places), the resulting expression is less convenient because we compute  on the  grid, and so it is easier to differentiate with respect to .

The  collision term is



## Output quantities

### Flux surface average:

For any quantity , the flux surface average can be computed from



where

.

Notice

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### Density perturbation

SFINCS returns the density carried in :

.

Upon flux surface averaging, we obtain



### Pressure perturbation

SFINCS also returns the pressure in , normalized to the reference pressure :



Upon flux surface averaging, we obtain



### Flow

We choose to normalize the parallel flow at each point as follows:

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Both numerical and analytic calculations often employ the weights average flow . In SFINCS, this quantity is normalized in the following way:

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### Particle flux

The following expression is useful for evaluating the radial fluxes:

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We may write the radial particle flux as



Using



then



where



### Momentum flux

We may write a radial momentum flux as



Using



then



where



### Heat flux

We may write the radial energy flux as



Using , then



where



### Parallel current

The current is normalized as follows:



so

.

We also form the average  which is associated with the bootstrap current:



where

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### Perturbed electrostatic potential

From eq (21) in the Physics of Plasmas paper, the variation of the electrostatic potential on a flux surface, , may be found from

.

We define the normalized perturbed potential to be , and so

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